MATHEMATICAL THEORY OF THE STATIONARY PROPAGATION VELOCITY OF A FINE-SCALE TURBULENT FLAME

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A boundary-value problem in the theory of propagation of a fine-scale turbulent flame is investigated, taking into account the influence of temperature and concentration pulsations on the magnitude of the heat liberation rate. In contrast to [1], the case when a second-order reaction proceeds in the flame is examined in detail. Conditions are found for the existence of a turbulent flame; the structure of the flame front is studied by a computational method. A change in the progress of the reaction is disclosed near the propagation limits.

1. Mathematical Formulation of the Problem

It has been shown in [1] that under the condition of equality of the laminar transport coefficient and their turbulent analogs and without taking account of the thermal expansion of the medium (constant density) the following boundary-value problem should be solved to find the turbulent combustion rate; given the equation

$$dp / du = \Phi / p - \omega_1 \quad (0<u<1) \tag{1.1}$$

and the boundary conditions

$$p(0) = 0, \quad p(1) = 0 \tag{1.2}$$

where

$$2\Phi = 2\Phi_\varepsilon = (u+Fp)^2 \exp\left[\frac{-\theta_0(u+Fp)}{1-\sigma(u+Fp)}\right] + (u-Fp)^2 \exp\left[\frac{-\theta_0(u-Fp)}{1-\sigma(u-Fp)}\right] \quad (0<u<\varepsilon)$$

$$2\Phi = 0 \quad (\varepsilon \leqslant u \leqslant 1)$$

$$F>0,\ \theta_0>0,\ 0<\sigma<1$$

Find the unique value $\omega_1>0$, if it exists, for which the solution of (1.1) will satisfy the conditions (1.2).

Here u, p, Φ, and ω_1 are respectively the dimensionless temperature, temperature gradient, mean chemical reaction rate, and velocity of flame propagation. The parameters F, θ_0, σ are known. The relation between the dimensionless and dimensional quantities is given in [1].

2. Existence and Uniqueness

Two solutions issue from the point (0,0), which is singular for (1.1). Only the positive solution for which

$$\frac{dp}{du}(0) = 0 \tag{2.1}$$

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which has physical meaning, will be examined below.

First, let us investigate the properties of the solution of the Cauchy problem

$$dp / du = \Phi_\varepsilon / p - \omega, \quad p(0) = 0 \tag{2.2}$$

The subscript in ω_1 will be omitted to cut down on writing.

Proposition 1. Let $(0, u_*)$ be the domain of uncontinuable solutions of (2.2). Then there exists a $k = k(u_*, \omega, F) > 0$, such that the inequality $p(u) - ku^2 > 0$ will be satisfied in the domain mentioned. The quantity k is here found just as in [1]:

$$k = \frac{1}{2u_*}\{-(\omega + AFu_*) + [(\omega + AFu_*)^2 + 2Au_*]^{1/2}\}$$

$$\left(A = \frac{1}{2}\exp\left[\frac{-\theta_0 u}{1 - \sigma u_*}\right]\right)$$

Exactly, as in [1], there follows from this proposition

$$\lim_{u \to u_*} (u + Fp) = 1/\sigma$$

Proposition 2. For every F there exist such ω, for which the domain of the uncontinuable solution $(0, u_*)$ of (2.2) is such that $u_* > \varepsilon$.

The inequality

$$\Phi_\varepsilon < (\varepsilon^2 + F^2p^2)\exp\left(\frac{\theta_0 Fp}{1 + \sigma Fp}\right) = \Phi_2(Fp)$$

is valid in the domain $p > 0$, $0 < u < \varepsilon$.

Hence, in the domain under consideration the solution of (2.2) will not exceed the solution of the equation

$$dp_1 / du = \Phi_2(Fp_1) / p_1 - \omega, \quad p_1(0) = 0$$

For sufficiently large ω, the equation

$$\Phi_2(Fp) - \omega p = 0 \tag{2.3}$$

has positive roots. Let p^+ be the lesser root of (2.3). Evidently, $p_1 < p^+$, and since p^+ tends monotonely to zero as ω increases, then starting with some value of ω, the inequality

$$p^+ \leqslant F^{-1}(\sigma^{-1} - \varepsilon)$$

is satisfied.

For those ω, for which the last inequality is true, $u_* > \varepsilon$. Indeed, if it is assumed that $u_* \leq \varepsilon$, then we obtain

$$\lim_{u \to u_*} (p - p^+) = \frac{1}{F}\left(\frac{1}{\sigma} - u_*\right) - p^+ \geqslant \frac{1}{F}\left(\frac{1}{\sigma} - u_*\right) - \frac{1}{F}\left(\frac{1}{\sigma} - \varepsilon\right) = \frac{1}{\sigma}(\varepsilon - u_*) \geqslant 0$$

which is impossible, since $p < p^+$.

Remark. A solution of the Cauchy problem (2.2) always exists in the domain (0,1) because of proposition 2.

When p (1) turns out to be positive for some ω, it is necessary to increase ω until the second condition of (1.2) is satisfied. When $p(1) = 0$, it is natural to diminish ω. It can hence turn out that p(u) reaches the line $u + Fp = 1/\sigma$ before p(1) vanishes, and, therefore, it is impossible to satisfy the second condition of (1.2). It will be shown below that this latter case is realized for F exceeding some limit F_* for any $0 < \sigma < 1$.

Let us consider the particular case $\sigma = 0$. The assumption that either the solution of the Cauchy problem (2.2) has a vertical asymptote for $u \le \varepsilon$, or $p(1) < 0$ if the solution exists in the whole domain $(0, \varepsilon]$, is false. Hence, for $\sigma = 0$ a solution of the boundary-value problem (1.1), (1.2) exists for any F and is, moreover, unique by virtue of the monotonicity of the function $p(1) = p(\omega, 1)$.

Proceeding to the proof of the existence of the limit value F_*, let us keep the inequality $0 < \sigma < 1$ in mind.

Let us show the existence of an F_1 such that for any $F \le F_1$ there exists a unique ω for which the solution of the boundary value problem (1.1), (1.2) exists.

To determine F_1, let us use the same system as in [1]

$$p^+(\varepsilon) = F^{-1}(\sigma^{-1} - \varepsilon), \quad p^-(\varepsilon) \geqslant \omega(1 - \varepsilon) \tag{2.4}$$

where $p^+(u)$ and $p^-(u)$ are defined in the domain $(0, \varepsilon)$, and

$$p^-(u) < p(u) < p^+(u)$$

Let us take the lower bound obtained in proposition 1 ($p^-(u) = ku^2$) as the function $p^-(u)$.

The solution to (2.3) yields the function p^+, but since this solution has not successfully been obtained explicitly, let us add the following equation:

$$\Phi_2(Fp^+) - \omega p^+ = 0$$

to the system (2.4).

Solving the system (2.4) in conjunction with the latter equation, we find F_1. Here and henceforth, when the estimates are considered only in the domain $(0, \varepsilon)$, then ε is substituted in place of u_* in the expression for A and k.

Now, let us show the existence of an F_2 such that for all $F \ge F_2$ no solution of (1.1) satisfying conditions (1.2) exists.

The system

$$F\omega = \frac{1/\sigma - \varepsilon}{1 - \varepsilon}, \qquad Fp^-(\varepsilon) \geqslant \frac{1}{\sigma} - \varepsilon \tag{2.5}$$

is used in [1] to find F_2.

Let $p^-(u) = ku^2$, then F_2 exists only when $\sigma > 2/3\varepsilon$ because the system (2.5) has a solution only for these σ.

Hence, to prove the existence of F_2 for all $0 < \sigma < 1$ below, a different method of constructing the lower bound is proposed.

If the new notation

$$Fp = \xi, \quad (\sigma^{-1} - \varepsilon) / (1 - \varepsilon) = \alpha$$

is introduced, then the problem of finding F_2 reduces to constructing the lower bound $\xi^-(u)$ for the equation

$$\frac{d\xi}{du} = F^2 \frac{\Phi_2(u, \xi)}{\xi} - \alpha, \qquad \xi(0) = 0 \tag{2.6}$$

such that the inequality

$$\xi^-(\varepsilon) \geqslant \sigma^{-1} - \varepsilon$$

would be satisfied.

Let us divide the domain $(0, \varepsilon)$ into two $(0, u_0)$ and (u_0, ε). Let us construct a function $\xi_1 = k_1 u^2$ in the domain $(0, u_0)$ such that k_1, u_0 would satisfy the inequality

$$1 - (\theta_0 + \sigma) u_0 (1 + k_1 u_0) > 0 \tag{2.7}$$

Furthermore, let us demand that k_1 satisfy the inequality

$$M \equiv 2F^2\Phi_\varepsilon (u, \xi) - 2\alpha\xi - 4k_1 u\xi \,|_{\xi=k_1u^2} > 0 \tag{2.8}$$

for all $u \in (0, u_0]$.

Taking (2.7) into account, we have

$$M > F^2u^2 \exp\left[\frac{-\theta_0(u+\xi)}{1-\sigma(u+\xi)}\right] - 2\alpha\xi - 4k_1u\xi\,|_{\xi=k_1u^2} \geqslant$$
$$\geqslant u^2\left\{F^2\exp\left[\frac{-\theta_0u_0(1+k_1u_0)}{1-\sigma u_0(1+k_1u_0)}\right] - 2\alpha k_1 - 4k_1^2u_0\right\} >$$
$$> u^2\left\{F^2\left[1 - \frac{\theta_0u_0(1+k_1u_0)}{1-\sigma u_0(1+k_1u_0)}\right] - 2\alpha k_1 - 4k_1^2u_0\right\} >$$
$$> u^2\{F^2[1-(\theta_0+\sigma)u_0(1+k_1u_0)] - 2\alpha k_1 - 4k_1^2u_0\}$$

The inequality (2.8) will be satisfied if k_1 is a positive root of the equation

$$F^2[1-(\theta_0+\sigma)u_0(1+k_1u_0)] - 2\alpha k_1 - 4k_1^2u_0 = 0$$

For convenience in the subsequent computations, let us take k_1 as a less awkward expression and less than the positive root of this equation

$$k_1 = \frac{F^2[1-(\theta_0+\sigma)u_0]}{2\alpha + 2F + F^2u_0^2(\theta_0+\sigma)}$$

Taking into account that in the domain (u_0, ε)

$$\Phi_\varepsilon > A(u-\xi)^2 > A(\xi^2 - 2u\xi)$$

let us construct the function

$$\xi_2(u) = \frac{\alpha+2}{AF^2} + 2u + \left[k_1u_0^2 - \left(2u_0 + \frac{\alpha+2}{AF^2}\right)\right]\exp[AF^2(u-u_0)]$$

which is a solution of the equation

$$\frac{d\xi_2}{du} = F^2\frac{A(\xi_2^2 - 2u\xi_2)}{\xi_2} - \alpha, \quad \xi_2(u_0) = k_1u_0^2$$

Let us select u_0 so that the inequality

$$\lim_{F\to\infty}\left[k_1u_0^2 - \left(2u_0 + \frac{\alpha+2}{AF^2}\right)\right] = \frac{1-3u_0(\theta_0+\sigma)}{\theta_0+\sigma} > 0$$

will be satisfied.

For example, let us take

$$u_0 < 1/6\,(\theta_0+\sigma)$$

Evidently there is an F_0 such that for $F \geq F_0$

$$k_1u_0^2 - \left(2u_0 + \frac{\alpha+2}{AF^2}\right) \geqslant \frac{1}{4(\theta_0+\sigma)} \tag{2.9}$$

Let us show that the function